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A STATE-VARIABLE APPROACH TO THE SOLUTION OF FREDHOLM INTEGRAL EQUATIONS

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M A S S A C H U S E T T S I N S T I T U T E O F T E C H N O L O G Y

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A STATE-VARIABLE APPROACH TO THE SOLUTION OF FREDHOLM INTEGRAL EQUATIONS

Arthur B. Baggeroer

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Abstract

A method of solving Fredholm integral equations of the second kind by state-variable techniques is presented. The principal advantage of this method is that it leads to efficient computer algorithms for calculating numerical solutions. The only assumptions that are made are (a) the kernel of the integral equation is the covariance function of a random process; (b) this random process is the output of a linear system having a white noise input; (c) this linear system has a finite dimensional state-variable description of its input-output relationship.

Both the homogeneous and nonhomogeneous integral equations are reduced to two linear first-order vector differential equations plus an associated set of boundary conditions. The coefficients of these differential equations follow directly from the matrices that describe the linear system. In the case of the homogeneous integral equation, the eigenvalues are found to be the solutions to a transcendental equation. The eigenfunctions also follow directly. In the case of the nonhomogeneous equation, the vector differential equations are identical to those obtained in the state-variable formulation of the optimum linear smoother. In both cases analytical and numerical examples are presented.

Finally, the optimum linear smoother (unrealizable filter) structure is derived by using a new approach. In this approach, the filter is required to be linear; then the resulting Wiener-Hopf equation is used in conjunction with techniques developed in the report to find the differential equations and boundary conditions specifying the optimum estimate.

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I. INTRODUCTION

In the theory of signal detection and estimation one frequently wishes to determine the solution to a Fredholm integral equation. These equations are of the form

$$\int_{T_o}^{T_f} \underline{K}_{\underline{y}}(t,\tau)\, \underline{g}(\tau)\, d\tau + \sigma \underline{g}(t) = \underline{s}(t), \qquad T_o \leq t \leq T_f, \tag{1}$$

where $\underline{K}_{\underline{y}}(t,\tau)$ is the kernel of the integral equation, $\underline{s}(t)$ is a known vector function, and $\underline{g}(t)$ is the desired solution.[1] In a communication theory context, the kernel $\underline{K}_{\underline{y}}(t,\tau)$ is the covariance matrix of a finite power vector, random process y(t); therefore, it is a symmetric, non-negative definite, square integrable function. (In this report, the treatment of vector random processes does not require any additional modifications to the theory for the scalar case.[1])

If $\underline{s}(t)$ is identically zero, Eq. 1 is called a homogeneous Fredholm integral equation. The homogeneous equation is usually written

$$\int_{T_o}^{T_f} \underline{K}_{\underline{g}}(t,\tau)\, \underline{\phi}(\tau)\, d\tau = \lambda \underline{\phi}(\tau), \qquad T_o \leq t \leq T_f. \tag{2}$$

In this case we have an eigenvalue problem. There are an at most countable number of values of $\lambda > 0$ for which solutions exist to Eq. 2, and there are no solutions for $\lambda < 0$. Equation 2 is probably most familiar in communication theory as the Karhunen-Loève expansion integral equation. (Note that we are using a vector eigenfunction-scalar eigenvalue expansion.[1])

If $K_y(t,\tau)$ is positive definite, then the solutions to Eq. 2 have positive eigenvalues and form a complete orthonormal set. However, if $K_y(t,\tau)$ is only non-negative definite, then there exists solutions $\underline{\phi}_o(t)$ with zero eigenvalues, i.e., they are orthogonal to the kernel

$$\int_{T_o}^{T_f} \underline{K}_{\underline{y}}(t,\tau)\, \underline{\phi}_o(\tau)\, d\tau = \underline{0}, \qquad T_o \leq t \leq T_f. \tag{3}$$

In this report we consider finding only those solutions with positive eigenvalues.

If $\underline{s}(t)$ is not identically zero and σ is nonzero (but not the negative of an eigenvalue of the homogeneous equation), then Eq. 1 is called a Fredholm equation of the second kind. This integral equation is used in communication theory for determining the optimal receiver structure for detecting a known signal in additive colored Gaussian noise. We may determine a conceptual solution in terms of the eigenfunctions of the homogeneous equation,

$$\underline{g}(t) = \frac{1}{\sigma}\left[\underline{s}(t) - \sum_i \frac{\lambda_i s_i}{\sigma + \lambda_i}\, \underline{\phi}_i(t)\right], \qquad T_o \leq t \leq T_f. \tag{4}$$

where

$$s_i = \int_{T_o}^{T_f} \underline{s}^T(\tau)\, \underline{\phi}_i(\tau)\, d\tau. \tag{5}$$

If σ is zero, i.e.,

$$\int_{T_o}^{T_f} K_{\underline{y}}(t,\tau)\, \underline{g}(\tau)\, d\tau = \underline{s}(t), \qquad T_o \leq t \leq T_f, \tag{6}$$

we have a Fredholm integral equation of the first kind. A solution does not in general exist unless one allows singularity functions. When a problem involves this type of equation, one may be led to singular detection or to an unstable test.[2] A survey for solving equations of this form has been given by Helstrom.[3]

In this report we want to consider solving Eqs. 1 and 2. When $K_{\underline{y}}(t,\tau)$ is stationary and has a rational Fourier transform, numerous techniques have been developed for solving these equations.[3-9] The approach which we use, however, is more general and has some distinct computational advantages.

We assume that $K_{\underline{y}}(t,\tau)$ is the covariance matrix of a random process $\underline{y}(t)$ which is the output of a linear system which has a finite dimensional state variable description and which is excited by white noise.[10-12] By using state variable methods we shall develop a systematic method which specifies a very general computational algorithm for solving integral equations with this type of kernel.

Before proceeding with our derivations, we shall mention our notation convention. Generally, scalars are lower case letters, not underscored; vectors are lower case letters, underscored; and matrices are upper case letters.

II. COVARIANCE FUNCTIONS FOR STATE-VARIABLE PROCESSES

We want to review some of the properties of the kernel $K_{\underline{y}}(t,\tau)$. The kernel $K_{\underline{y}}(t,\tau)$ is the covariance matrix of a random process $\underline{y}(t)$ that is obtained by passing white noise through a linear system whose dynamics have an n-dimensional state-variable description.

Let us define this state-variable system as follows:

$$\frac{d\underline{x}(t)}{dt} = F(t)\,\underline{x}(t) + G(t)\,\underline{u}(t) \qquad \text{(linear state equation)}, \tag{7}$$

$$\underline{y}(t) = C(t)\,\underline{x}(t) \qquad \text{(linear observation equation)}. \tag{8}$$

The covariance matrix of the zero-mean white process $\underline{u}(t)$ is

$$E[\underline{u}(t)\,\underline{u}^T(\tau)] = Q\delta(t-\tau). \tag{9}$$

Let us also consider the initial state of the system to be a zero-mean random vector with

$$E[\underline{x}(T_o)\,\underline{x}^T(T_o)] = P_o. \tag{10}$$

This completes our assumptions regarding the statistics which are involved.

By using Eq. 8, the covariance matrix $K_{\underline{y}}(t,\tau)$ is easily related to the covariance matrix of the state vector $\underline{x}(t)$

$$K_{\underline{y}}(t,\tau) = C(t)\,K_{\underline{x}}(t,\tau)\,C^T(\tau). \tag{11}$$

One can show (see, for example, Van Trees[1]) that

$$K_{\underline{x}}(t,\tau) = \begin{cases} \Theta(t,\tau)\,K_{\underline{x}}(\tau,\tau) & \text{for } t \geq \tau \\ \\ K_{\underline{x}}(t,t)\,\Theta^T(\tau,t) & \text{for } \tau \geq t, \end{cases} \tag{12}$$

where $\Theta(t,\tau)$ is the transition matrix for the system defined by the matrix $F(t)$. That is,

$$\frac{\partial}{\partial t}\Theta(t,\tau) = F(t)\,\Theta(t,\tau) \tag{13a}$$

$$\Theta(\tau,\tau) = I. \tag{13b}$$

Furthermore, it may also be shown that the matrix $K_{\underline{x}}(t,t)$ satisfies the following differential equation[1]

$$\frac{d}{dt}K_{\underline{x}}(t,t) = F(t)\,K_{\underline{x}}(t,t) + K_{\underline{x}}(t,t)\,F^T(t) + G(t)\,Q\,G^T(t) \tag{14a}$$

with the initial condition

$$K_{\underline{x}}(T_o,T_o) = P_o. \tag{14b}$$

III. DERIVATION OF THE SOLUTION TECHNIQUE

Both the homogeneous and nonhomogeneous equations involve an integral operation of the form

$$\underline{\zeta}(t) = \int_{T_o}^{T_f} K_{\underline{y}}(t,\tau)\, \underline{f}(\tau)\, d\tau, \qquad T_o \leq t \leq T_f, \tag{15}$$

where $\underline{f}(t)$ is related to the eigenfunction $\underline{\phi}(t)$ in the homogeneous equation or is the solution $\underline{g}(t)$ in the nonhomogeneous equation. By using Eq. 11, we may write this operation as

$$\underline{\zeta}(t) = C(t)\, \underline{\xi}(t), \tag{16}$$

where

$$\underline{\xi}(t) \triangleq \int_{T_o}^{T_f} K_{\underline{x}}(t,\tau)\, C^T(\tau)\, \underline{f}(\tau)\, d\tau, \qquad T_o \leq t \leq T_f. \tag{17}$$

We shall now determine a set of differential equations for the function $\underline{\xi}(t)$. Solving these differential equations is equivalent to performing the integral operation on Eq. 17. By using these results, we shall be able to solve both the homogeneous and nonhomogeneous equations by choosing $\underline{f}(t)$ appropriately.

Let us now proceed with our derivation of the differential equations for $\underline{\xi}(t)$. By substituting Eq. 12 in Eq. 17, we have

$$\underline{\xi}(t) = \int_{T_o}^{t} \Theta(t,\tau)\, K_{\underline{x}}(\tau,\tau)\, C^T(\tau)\, \underline{f}(\tau)\, d\tau$$

$$+ K_{\underline{x}}(t,t) \int_{t}^{T_f} \Theta^T(\tau,t)\, C^T(\tau)\, \underline{f}(\tau)\, d\tau, \qquad T_o \leq t \leq T_f. \tag{18}$$

If we differentiate Eq. 18 with respect to t, we obtain

$$\frac{d\underline{\xi}(t)}{dt} = \int_{T_o}^{t} \frac{\partial\Theta(t,\tau)}{\partial t}\, K_{\underline{x}}(\tau,\tau)\, C^T(\tau)\, \underline{f}(\tau)\, d\tau$$

$$+ \frac{dK_{\underline{x}}(t,t)}{dt} \int_{t}^{T_f} \Theta^T(\tau,t)\, C^T(\tau)\, \underline{f}(\tau)\, d\tau$$

$$+ K_{\underline{x}}(t,t) \int_{t}^{T_f} \frac{\partial\Theta^T(\tau,t)}{\partial t}\, C^T(\tau)\, \underline{f}(\tau)\, d\tau, \qquad T_o \leq t \leq T_f \tag{19}$$

in which we have used Eq. 13b and cancelled two equal terms. In the first term of the right-hand side of Eq. 19 we substitute Eq. 13a, and in the last term we use the fact that $\theta^T(\tau,t)$ is the transition matrix for the adjoint equation of the matrix $F(t)$.[11] That is,

$$\frac{\partial}{\partial t}\,\theta^T(\tau,t) = -F(t)\,\theta^T(\tau,t). \tag{20}$$

When we make these two substitutions in Eq. 19, we obtain

$$\begin{aligned}\frac{d\underline{\xi}(t)}{dt} &= F(t)\int_{T_o}^{t}\theta(t,\tau)\,\underline{K_x}(\tau,\tau)\,C^T(\tau)\,\underline{f}(\tau)\,d\tau\\ &\quad+\left[\frac{d\underline{K_x}(t,t)}{dt} - \underline{K_x}(t,t)\,F^T(t)\right]\int_{t}^{T_f}\theta^T(\tau,t)\,C^T(\tau)\,\underline{f}(\tau)\,d\tau,\\ &\qquad T_o \leqslant t \leqslant T_f.\end{aligned} \tag{21}$$

By applying (14a), we obtain

$$\begin{aligned}\frac{d\underline{\xi}(t)}{dt} &= F(t)\int_{T_o}^{t}\theta(t,\tau)\,\underline{K_x}(\tau,\tau)\,C^T(\tau)\,\underline{f}(\tau)\,d\tau\\ &\quad+\left[F(t)\,\underline{K_x}(t,t) + G(t)\,Q\,G^T(t)\right]\int_{t}^{T_f}\theta^T(\tau,t)\,C^T(\tau)\,\underline{f}(\tau)\,d\tau,\\ &\qquad T_o \leqslant t \leqslant T_f.\end{aligned} \tag{22}$$

After rearranging terms and using Eq. 18, we finally have

$$\frac{d\underline{\xi}(t)}{dt} = F(t)\,\underline{\xi}(t) + G(t)\,Q\,G^T(t)\int_{t}^{T_f}\theta^T(\tau,t)\,C^T(\tau)\,\underline{f}(\tau)\,d\tau, \quad T_o \leqslant t \leqslant T_f. \tag{23}$$

At this point we have derived a differential equation for $\underline{\xi}(t)$; however, we see that an integral operation still remains. Let us simply define this integral operation as a second linear functional of $\underline{f}(t)$:

$$\underline{\eta}(t) \triangleq \int_{t}^{T_f}\theta^T(\tau,t)\,C^T(\tau)\,\underline{f}(\tau)\,d\tau, \qquad T_o \leqslant t \leqslant T_f. \tag{24}$$

Therefore, we have

$$\frac{d\underline{\xi}(t)}{dt} = F(t)\,\underline{\xi}(t) + G(t)\,Q\,G^T(t)\,\underline{\eta}(t), \qquad T_o \leqslant t \leqslant T_f. \tag{25}$$

It is now a simple matter to derive a second differential equation which $\underline{\eta}(t)$ satisfies. Differentiating Eq. 24 gives us

$$\frac{d\underline{\eta}(t)}{dt} = -C^T(t)\,\underline{f}(t) - F^T(t)\int_t^{T_f} \theta^T(\tau,t)\,C^T(\tau)\,\underline{f}(\tau)\,d\tau, \qquad T_o \leq t \leq T_f \tag{26}$$

where we have again used the adjoint relationship given by Eq. 20. After substituting Eq. 22, we have

$$\frac{d\underline{\eta}(t)}{dt} = -C^T(t)\,\underline{f}(t) - F^T(t)\,\underline{\eta}(t), \qquad T_o \leq t \leq T_f. \tag{27}$$

We now want to derive two sets of boundary conditions which Eqs. 25 and 27 satisfy. For both the homogeneous and nonhomogeneous equations ($\sigma > 0$), $\underline{f}(t)$ must be bounded at the end points, $t = T_o$ and $t = T_f$. Consequently, by setting $t = T_f$ in Eq. 24, we obtain

$$\underline{\eta}(T_f) = \underline{0}. \tag{28}$$

The second boundary condition follows directly from Eq. 18. If we set $t = T_o$, the first term is zero, while the second term may be written

$$\underline{\xi}(T_o) = K_{\underline{x}}(T_o,T_o)\int_{T_o}^{T_f} \theta^T(\tau,t)\,C(\tau)\,\underline{f}(\tau)\,d\tau, \tag{29}$$

or,

$$\underline{\xi}(T_o) = K_{\underline{x}}(T_o,T_o)\,\underline{\eta}(T_o) = P_o\underline{\eta}(T_o). \tag{30}$$

It is easy to see that the two boundary conditions given by Eqs. 28 and 30 are independent.

We may now summarize the results of our derivation. We have derived two differential equations (Eqs. 25 and 27):

$$\frac{d\underline{\xi}(t)}{dt} = F(t)\,\underline{\xi}(t) + G(t)\,QG^T(t)\,\underline{\eta}(t), \qquad T_o \leq t \leq T_f$$

$$\frac{d\underline{\eta}(t)}{dt} = -C(t)\,\underline{f}(t) - F^T(t)\,\underline{\eta}(t), \qquad T_o \leq t \leq T_f.$$

In addition, we have the boundary conditions (Eqs. 30 and 28):

$$P_o\underline{\eta}(T_o) = \underline{\xi}(T_o),$$

$$\underline{\eta}(T_f) = \underline{0}.$$

Notice that the only property of $\underline{f}(t)$ which we required was its boundedness at the endpoints of the interval. (This excludes considering equations of the first kind where singularity functions may appear there.) Equations 28 and 30 each imply n linearly independent boundary conditions. Since the differential equations are linear, any solution that satisfies the boundary conditions is unique. Finally, the derivation of these equations can be reversed in order to obtain the functional defined by Eq. 17; that is, we can integrate the differential equations rather than differentiate the integral equation. Consequently, the solution $\underline{\xi}(t)$ to the differential equations must be identical to the result of the functional operation of Eq. 17. This implies that the existence of a solution $\underline{\xi}(t)$ which satisfies the boundary conditions is necessary and sufficient for the existence of a solution to Eq. 17.

We now want to use these differential equations to solve the homogeneous and nonhomogeneous Fredholm equations of the second kind.

IV. HOMOGENEOUS FREDHOLM EQUATIONS

We shall now use the results of the previous section to determine a method for finding the eigenvalues and eigenfunctions for homogeneous Fredholm equations when the kernel, $K_{\underline{y}}(t,\tau)$ is the covariance matrix of the output of a linear system that is described in terms of state variables and is excited by white noise.

The integral equation for this problem is Eq. 2:

$$\int_{T_o}^{T_f} K_{\underline{y}}(t,\tau)\, \underline{\phi}_i(\tau)\, d\tau = \lambda_i \underline{\phi}_i(t), \qquad T_o \leq t \leq T_f.$$

When we employ Eq. 11, we may write (2) as

$$C(t) \left(\int_{T_o}^{T_f} K_{\underline{x}}(t,\tau)\, C^T(\tau)\, \underline{\phi}_i(\tau)\, d\tau \right) = \lambda_i \underline{\phi}_i(t), \qquad T_o \leq t \leq T_f. \tag{31}$$

Let us now put (31) into such a form that we can employ our previous results. If in Eq. 31 we set

$$\underline{\phi}_i(t) = \underline{f}(t), \tag{32}$$

the result is that the integral enclosed by parentheses is the function $\underline{\xi}(t)$ as defined in Eq. 17. Consequently, let us define $\underline{\xi}_i(t)$ to be

$$\underline{\xi}_i(t) = \int_{T_o}^{T_f} K_{\underline{x}}(t,\tau)\, C^T(\tau)\, \underline{\phi}_i(\tau)\, d\tau, \qquad T_o \leq t \leq T_f; \tag{33}$$

so that (31) becomes

$$C(t)\, \underline{\xi}_i(t) = \lambda_i \underline{\phi}_i(t), \qquad T_o \leq t \leq T_f. \tag{34}$$

If we assume that λ_i is positive, which is guaranteed if $K_{\underline{y}}(t,\tau)$ is positive definite, we can solve for the eigenfunction in terms of $\underline{\xi}_i(t)$. This gives us

$$\underline{\phi}_i(t) = \frac{1}{\lambda_i}\, C(t)\, \underline{\xi}_i(t), \qquad T_o \leq t \leq T_f. \tag{35}$$

We have found an equivalent differential equation representation for the functional defined by Eq. 33. If in Eq. 17 we set

$$\underline{f}(t) = \underline{\phi}_i(t) = \frac{1}{\lambda_i}\, C(t)\, \underline{\xi}_i(t), \qquad T_o \leq t \leq T_f, \tag{36}$$

we can reduce (33) to a set of differential equations with an associated two-point boundary coundition upon their solution. From (25) and (27) the differential equations are

$$\frac{d}{dt}\,\underline{\xi}_i(t) = F(t)\,\underline{\xi}_i(t) + G(t)\,Q\,G^T(t)\,\underline{\eta}_i(t), \qquad T_o \leq t \leq T_f, \tag{37}$$

$$\frac{d}{dt}\,\underline{\eta}_i(t) = -\frac{C^T(t)\,C(t)}{\lambda_i}\,\underline{\xi}_i(t) - F^T(t)\,\underline{\eta}_i(t), \qquad T_o \leq t \leq T_f. \tag{38}$$

From (26) and (28), the boundary conditions are

$$\underline{\eta}_i(T_f) = \underline{0}, \tag{39}$$

$$\underline{\xi}_i(T_o) = P_o\underline{\eta}_i(T_o). \tag{40}$$

The desired eigenfunction is given by Eq. 35.

The result of Eqs. 31-40 is that we have transformed the homogeneous Fredholm integral equation into a set of differential equations whose coefficients are directly related to the state equations and covariance matrices that are used to generate the random process $\underline{y}(t)$.

Notice that we have a set of 2n differential equations to solve. This is consistent with previous methods for treating stationary processes. In these methods, one has a 2n-order differential equation to solve, where 2n is the degree of the denominator polynomial of the spectrum.[1]

Equation 35 implies that all of the solutions to Eq. 2 with positive λ are contained in the range space defined by C(t). We should note that if C(t) is not onto for a set of t with nonzero measure, then $K_{\underline{y}}(t,\tau)$ is not positive definite. Therefore, for this case there may be solutions with λ equal to zero which are not contained in this range space.

We now want to determine the eigenvalues and eigenfunctions for (2). Let us define the (2n x 2n) matrix $W(t:\lambda)$ as

$$W(t:\lambda) = \left[\begin{array}{c|c} F(t) & G(t)\,Q\,G^T(t) \\ \hline -\dfrac{C^T(t)\,C(t)}{\lambda} & -F^T(t) \end{array}\right], \tag{41}$$

so that in vector form Eqs. 37 and 38 become

$$\frac{d}{dt}\left[\begin{array}{c} \underline{\xi}_i(t) \\ \hline \underline{\eta}_i(t) \end{array}\right] = W(t:\lambda_i)\left[\begin{array}{c} \underline{\xi}_i(t) \\ \hline \underline{\eta}_i(t) \end{array}\right], \qquad T_o \leq t \leq T_f. \tag{42}$$

Furthermore, let us define the transition matrix associated with $W(t:\lambda)$ by

$$\frac{\partial}{\partial t}\,\Psi(t,T_o:\lambda) = W(t:\lambda)\;\Psi(t,T_o:\lambda), \tag{43}$$

$$\Psi(T_o,T_o:\lambda) = I. \tag{44}$$

We have emphasized the λ dependence of $W(t:\lambda)$ and $\Psi(t,T_o:\lambda)$ by including λ as an argument.

In terms of this transition matrix, the most general solution to Eq. 42 is

$$\begin{bmatrix} \underline{\xi}_i(t) \\ \hline \underline{\eta}_i(t) \end{bmatrix} = \Psi(t,T_o:\lambda_i)\begin{bmatrix} \underline{\xi}_i(T_o) \\ \hline \underline{\eta}_i(T_o) \end{bmatrix}. \tag{45}$$

After employing the boundary condition specified by Eq. 40, we have

$$\begin{bmatrix} \underline{\xi}_i(t) \\ \hline \underline{\eta}_i(t) \end{bmatrix} = \Psi(t,T_o:\lambda_i)\begin{bmatrix} P_o \\ \hline I \end{bmatrix}\underline{\eta}_i(T_o). \tag{46}$$

Let us now partition $\Psi(t,T_o:\lambda)$ into four n by n matrices such that

$$\Psi(t,T_o:\lambda) \triangleq \left[\begin{array}{c|c} \Psi_{\underline{\xi}\underline{\xi}}(t,T_o:\lambda) & \Psi_{\underline{\xi}\underline{\eta}}(t,T_o:\lambda) \\ \hline \Psi_{\underline{\eta}\underline{\xi}}(t,T_o:\lambda) & \Psi_{\underline{\eta}\underline{\eta}}(t,T_o:\lambda) \end{array}\right]. \tag{47}$$

Rewriting Eq. 41, in terms of these partitions, we have

$$\begin{bmatrix} \underline{\xi}_i(t) \\ \hline \underline{\eta}_i(t) \end{bmatrix} = \begin{bmatrix} \Psi_{\underline{\xi}\underline{\xi}}(t,T_o:\lambda_i)\;P_o + \Psi_{\underline{\xi}\underline{\eta}}(t,T_o:\lambda_i) \\ \hline \Psi_{\underline{\eta}\underline{\xi}}(t,T_o:\lambda_i)\;P_o + \Psi_{\underline{\eta}\underline{\eta}}(t,T_o:\lambda_i) \end{bmatrix}\underline{\eta}_i(T_o). \tag{48}$$

The boundary condition given by Eq. 39 requires

$$\underline{\eta}_i(T_f) = [\Psi_{\underline{\eta}\underline{\xi}}(T_f,T_o:\lambda_i)\;P_o + \Psi_{\underline{\eta}\underline{\eta}}(T_f,T_o:\lambda_i)]\underline{\eta}_i(T_o) = 0. \tag{49}$$

This implies one of two consequences. Either

$$\underline{\eta}_i(T_o) = \underline{0} \tag{50}$$

which implies a trivial zero solution; or,

$$\det[\Psi_{\underline{\eta}\underline{\xi}}(T_f,T_o:\lambda_i)\;P_o + \Psi_{\underline{\eta}\underline{\eta}}(T_f,T_o:\lambda_i)] = 0. \tag{51}$$

If the latter is true, Eq. 47 has a nontrivial solution which satisfies the requisite boundary conditions. Because of the functional equivalence of these differential equations and the original integral equation, this nontrivial solution to Eq. 42 implies that λ_i is an eigenvalue. That is, the eigenvalues of Eq. 2 are simply those values of λ_i that satisfy Eq. 51.

For covenience, define $A(\lambda)$ as

$$A(\lambda) = \underline{\Psi}_{\eta\xi}(T_f,T_o:\lambda)\ P_o + \underline{\Psi}_{\eta\eta}(T_f,T_o:\lambda). \tag{52}$$

When λ is equal to an eigenvalue, λ_i, $A(\lambda_i)$ has a vanishing determinant. Consequently, the characteristic polynomial of $A(\lambda_i)$ has a root equal to zero and $\underline{\eta}_i(T_o)$ is the characteristic vector associated with this root. (We have used the adjective "characteristic" in order to avoid confusing the eigenvalue properties of the matrix $A(\lambda_i)$ with those of the integral equation, Eq. 2.)

> [Note: The function $\det(A(\lambda))$ has an interesting interpretation. Since it vanishes only at the eigenvalues, we are able to relate it to the Fredholm determinant.[6] The Fredholm determinant is given by
>
> $$D(z) = \prod_{i=1}^{\infty} (1 + \lambda_i z),$$
>
> which vanishes for $z = -1/\lambda_i$. Consequently, we must have
>
> $$D(z) = A_o \det(A(-1/z)),$$
>
> where A_o is a multiplicative constant which may be determined by
>
> $$1/A_o = \lim_{\lambda \to \infty} \det(A(\lambda)).]$$

We can now summarize our results for homogeneous Fredholm equations. The eigenvalues λ_i are specified by the roots of the transcendental equation

$$\det A(\lambda_i) = 0, \tag{53}$$

where $A(\lambda)$ is given by Eq. 52. Within a multiplicative factor, the eigenfunctions are given by

$$\underline{\phi}_i(t) = \frac{C(t)}{\lambda_i}\left[\underline{\Psi}_{\xi\xi}(t,T_o:\lambda_i)\ P_o + \underline{\Psi}_{\xi\eta}(t,T_o:\lambda_i)\right] \underline{\eta}_i(T_o), \tag{54}$$

where $\underline{\eta}_i(T_o)$ satisfies the orthogonality relationship

$$A(\lambda_i)\ \underline{\eta}_i(T_o) = \underline{0}. \tag{55}$$

The multiplicative factor may be determined by applying the normality requirement. The matrices

$$\underline{\Psi}_{\xi\xi}(t,T_o:\lambda),\ \underline{\Psi}_{\xi\eta}(t,T_o:\lambda),\ \underline{\Psi}_{\eta\xi}(T_f,T_o:\lambda), \text{ and } \underline{\Psi}_{\eta\eta}(T_f,T_o:\lambda)$$

are partitions of the matrix $\Psi(t,T_o:\lambda)$ which is the transition matrix associated with the matrix

$$W(t:\lambda) = \left[\begin{array}{c|c} F(t) & G(t)\,Q\,G^T(t) \\ \hline -\dfrac{C^T(t)\,C(t)}{\lambda} & -F^T(t) \end{array}\right]. \tag{41}$$

These equations specify the eigenvalues and eigenfunctions for a kernel $K_{\underline{y}}(t,\tau)$ which is the covariance matrix for the output $\underline{y}(t)$ of a linear, possibly time-varying system with a white noise excitation.

Before proceeding we should comment about multiple-order roots. In general, the function det $A(\lambda)$ vanishes with nonzero slope, that is, near an eigenvalue λ_i,

$$\det A(\lambda) = c_1(\lambda-\lambda_i) + c_2(\lambda-\lambda_i)^2 + \ldots, \tag{56a}$$

where c_1 is nonzero. In the case of multiple-order eigenvalues however, the function det $(A(\lambda))$ vanishes tangentially; that is, near an eigenvalue λ_i of order ℓ

$$\det A(\lambda) = c_\ell(\lambda-\lambda_i)^\ell + c_{\ell+1}(\lambda-\lambda_i)^{\ell+1} + \ldots . \tag{56b}$$

This implies that there will be ℓ linearly independent vectors $\underline{\eta}_i(T_o)$ satisfying

$$A(\lambda_i)\,\underline{\eta}_i(T_o) = 0. \tag{55}$$

Let us pause briefly now to point out some advantages that this technique offers.

1. We can solve Eq. 2 in the vector case. For those techniques which rely upon spectral factorization methods the vector case could cause some difficulty.

2. Kernels for certain time-varying systems may be studied by using numerical methods.

3. Once the state matrices are chosen to generate $\underline{y}(t)$, the equations that must be solved follow directly.

4. One does not have to substitute any functions back into the original integral equation in order to determine the transcendental equation that determines the eigenvalues.

5. We can solve for each eigenvalue and eigenfunction independently of the others, which is significant in actually obtaining accurate solutions.

6. Finally, the most important advantage is that the technique is very well suited to numerical methods. This allows one to determine numerical solutions easily for problems in which an analytic calculation is either difficult or not feasible.

The fundamental issue which determines whether one can obtain analytic solutions is the calculation of the transition matrix $\Psi(t,T_o:\lambda)$. In the case of time-varying systems, no general analytic method exists for determining this matrix. Consequently, one cannot in general find an analytic expression for Eq. 51 and the corresponding eigenfunctions. One can still, however, apply numerical methods to calculate this transition matrix.

For kernels that are the covariances of constant parameter systems, one can always find an analytic expression for this transition matrix. In this case we have

$$\Psi(t,T_o:\lambda) = e^{W(\lambda)[t-T_o]}, \tag{57a}$$

where we have employed a matrix exponential. For analytic solutions this matrix exponential may be conveniently computed by Laplace transform techniques. It is easy to show that

$$e^{W(\lambda)t} = \mathscr{L}^{-1}\{[Is-W(\lambda)]^{-1}\}, \tag{57b}$$

where $\mathscr{L}^{-1}$ is the inverse Laplace operator. In the inversion, the contour must be taken to the right of all pole locations of $[Is-W(\lambda)]^{-1}$. If one desires a numerical evaluation of this matrix exponential, one can truncate the series expansion

$$e^{W(\lambda)t} = \sum_{j=0}^{\infty} \frac{[W(\lambda)]^j \, t^j}{j!} \, . \tag{57c}$$

The analysis of constant parameter systems is particularly important when one wishes to study stationary kernels, that is, $K_{\underline{y}}(t,\tau) = K_{\underline{y}}(t - \tau)$. In order to study this class of problems, one uses a constant parameter system with P_o chosen appropriately, in particular

$$P_o = \lim_{t \to \infty} K_{\underline{x}}(t,t). \tag{58}$$

V. EXAMPLES OF EIGENVALUE AND EIGENFUNCTION DETERMINATION

We shall now illustrate the method developed in Section IV by considering some examples. We shall do two examples analytically for first-order processes. We do this principally to illustrate the use of the formulas. We shall then present an example of the numerical analysis of a second-order system. It is this type of problem for which the technique is most useful. It allows one to obtain numerical solutions very quickly with a digital computer.

<u>Example 1</u>

The first example that we want to consider is the Wiener process. The covariance matrix for this process is

$$K_y(t,\tau) = \mu^2 \min(t,\tau), \qquad 0 \leq t,\tau. \tag{59}$$

A state-variable representation of a system which generates y(t) is

$$\frac{dx(t)}{dt} = u(t), \qquad 0 \leqslant t \leqslant T \qquad \text{(state equation)} \tag{60a}$$

$$y(t) = \mu x(t) \qquad \text{(observation equation)}, \tag{60b}$$

where

$$E[u(t)\, u(\tau)] = \delta(t-\tau) \tag{60c}$$

$$E[x^2(0)] = 0. \tag{60d}$$

The matrices involved are

$$\begin{aligned} F &= 0 \qquad & C &= \mu \\ G &= 1 & P_o &= 0 \\ Q &= 1 & & . \end{aligned} \tag{61a-61e}$$

The matrix $W(\lambda)$ is obtained by substituting (61) in (41),

$$W(\lambda) = \left[\begin{array}{c|c} 0 & 1 \\ \hline -\frac{\mu^2}{\lambda} & 0 \end{array}\right]. \tag{62}$$

By applying Eq. 57, we find that the transition matrix $\Psi(t,0:\lambda)$ is

$$\Psi(t,0:\lambda) = \left[\begin{array}{c|c} \cos\left(\frac{\mu}{\sqrt{\lambda}} t\right) & \frac{\sqrt{\lambda}}{\sigma} \sin\left(\frac{\mu}{\sqrt{\lambda}} t\right) \\ \hline -\frac{\mu}{\sqrt{\lambda}} \sin\left(\frac{\mu}{\sqrt{\lambda}} t\right) & \cos\left(\frac{\mu}{\sqrt{\lambda}} t\right) \end{array}\right]. \tag{63}$$

In order for an eigenvalue to exist, Eqs. 52 and 53 imply

$$\det(A(\lambda_i)) = \cos\left(\frac{\mu}{\sqrt{\lambda_i}} T\right) = 0. \tag{64}$$

This is satisfied when

$$\lambda_i = \left[\frac{2\mu T}{(2i+1)\pi}\right]^2 \tag{65}$$

$i = 0, 1, 2.$

The eigenfunctions follow by substituting Eq. 63 in Eq. 54. After determining the appropriate normalization factor, we have

$$\phi_i(t) = \sqrt{\frac{2}{T}} \sin\left[\left(\frac{2i+1}{2}\right)\frac{\pi}{T} t\right] \qquad 0 \leqslant t \leqslant T. \tag{66}$$

Example 2a

Let us now consider our kernel to be the covariance function for a stationary random process which is the output of a first-order system. The covariance function for this process is

$$K_y(t,\tau) = Se^{-k|t-\tau|}. \tag{67}$$

The state equations that generate this process are

$$\frac{d\underline{x}(t)}{dt} = -kx(t) + u(t) \qquad 0 \leqslant t \leqslant T \qquad \text{(state equation)} \tag{67a}$$

$$y(t) = x(t) \qquad \text{(observation equation)} \tag{67b}$$

$$E[u(t)\, u(\tau)] = 2kS\delta(t-\tau)$$

$$E[x^2(0)] = S, \qquad E[x(0)] = 0.$$

The matrices involved are

$$F = -k \qquad C = 1$$

$$G = 1 \qquad P_o = S \tag{68a-68e}$$

$$Q = 2kS$$

The matrix $W(\lambda)$ follows by substituting Eqs. 68a-68d in (41)

$$W(\lambda) = \left[\begin{array}{c|c} -k & 2kS \\ \hline -\frac{1}{\lambda} & k \end{array}\right] . \tag{69}$$

The transition matrix for $W(\lambda)$ is

$$\Psi(t,0:\lambda) = \left[\begin{array}{c|c} \cos(kbt) - \frac{\sin(kbt)}{b} & \frac{2P}{b}\sin(kbt) \\ \hline -\frac{1}{\lambda k}\frac{\sin(kbt)}{b} & \cos(kbt) + \frac{\sin(kbt)}{b} \end{array}\right] , \tag{70}$$

where

$$b \triangleq \sqrt{\frac{2S}{k\lambda} - 1} . \tag{71}$$

By substituting Eq. 70 in (52) and (53), we obtain an equation which determines our eigenvalues.

$$\det(A(\lambda_i)) = \frac{1}{b_i} [1 - \frac{S}{\lambda_i k}] \sin(kb_i T) + \cos(kb_i T) = 0 \tag{72}$$

In order to compute the roots by hand, Eq. 72 can be put in a more convenient form. This form is

$$\tan(kb_i T) = \frac{2b_i}{b_i^2 - 1} . \tag{73}$$

Solving Eq. 71 for λ gives us the expression for the eigenvalues, λ_i, in terms of the b_i,

$$\lambda_i = \frac{2S}{k} \left[1+b_i^2\right]^{-1} . \tag{74}$$

Applying (54) gives us the eigenfunctions. They are of the form

$$\phi_i(t) = \gamma_i[\cos(kb_i t) + \frac{1}{b_i} \sin(kb_i t)], \qquad 0 \leq t \leq T, \tag{75}$$

where γ_i is a normalizing factor.

Example 2b

The output process of a constant parameter system is not necessarily stationary. An example of this can be generated from the previous example. Instead of setting P_o equal to the mean-square power of the stationary process, assume that we know the state at $t = 0$ exactly; that is,

$$P_o = 0. \tag{76}$$

In this case the covariance function becomes

$$K_y(t,\tau) = \begin{cases} S\, e^{-kt}(e^{k\tau} - e^{-k\tau}) & \text{for } t > \tau \\ S\, e^{-k\tau}(e^{kt} - e^{-kt}) & \text{for } \tau > t \end{cases} \tag{77}$$

By substituting Eq. 70 in (52) and setting P_o equal to zero, the equation for the eigenvalues follows directly:

$$\det(A(\lambda_i)) = \cos(kb_iT) + \frac{\sin(kb_iT)}{b_i} = 0, \tag{78}$$

where, as before,

$$b = \sqrt{\frac{2S}{k\lambda} - 1}$$

or equivalently,

$$\tan(kb_iT) = -b_i. \tag{79}$$

From (54) and (70), the eigenfunctions have the form

$$\phi_i(t) = \gamma_i \sin(kb_it) \qquad 0 \leq t \leq T, \tag{80}$$

where γ_i is again a normalizing factor.

Example 3

In this example we want to consider the analysis of a second-order system. In contrast with the two previous examples, however, we shall consider a particular system and analyze it by using numerical methods. Obtaining analytic results for systems whose dimension is greater than one is straightforward, but extremely tedious.

Let us consider a random process y(t) which is generated by a system with the following set of state equations:

$$F = \begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad P_o = \begin{bmatrix} 4 & 0 \\ 0 & 40 \end{bmatrix} \tag{81a-81e}$$

$$Q = 160$$

The covariance function of the observed signal y(t), for this choice of the matrix P_o, is

$$K_y(t,\tau) = \frac{4}{3} e^{-|t-\tau|} [3 \cos (3|t-\tau|) + \sin 3|t-\tau|], \tag{82}$$

which is plotted in Fig. 1a. The corresponding spectrum is

$$S_y(\omega) = \frac{160}{\omega^4 - 16\omega^2 + 100}, \tag{83}$$

which is plotted in Fig. 1b.

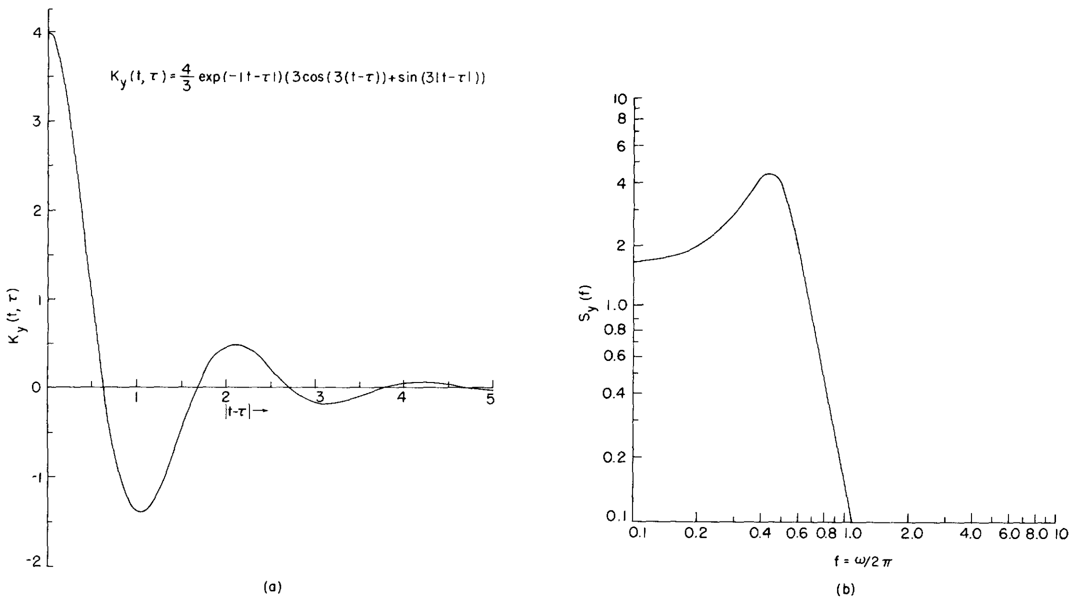

Fig. 1. (a) Covariance function for a second-order system.
(b) Spectrum for a second-order system.

In order to find the eigenvalues for this particular covariance function, we need to plot $\det(A(\lambda))$ against λ, where the matrix $A(\lambda)$ is given by Eq. 52. We have done this in Fig. 2, where we have chosen $T_f - T_o = 2$. In this figure the zero crossings are the desired eigenvalues. This type of behavior for the function $\det(A(\lambda))$ is typical of those that we have observed. In the region of the larger eigenvalues, corresponding to those with significant energy, the function is well-behaved and oscillating (nonperiodically). As we approach the less significant eigenvalues, however, the amplitude of the oscillation rapidly increases. Eventually, the eigenvalues become so small that it becomes difficult to compute $A(\lambda)$ accurately. In this region the eigenvalues are approaching their asymptotic behavior, as discussed by Capon.[13]

Since this state-variable technique is well suited for finding the significant eigenvalues, one could combine this method with an asymptotic method in order to find

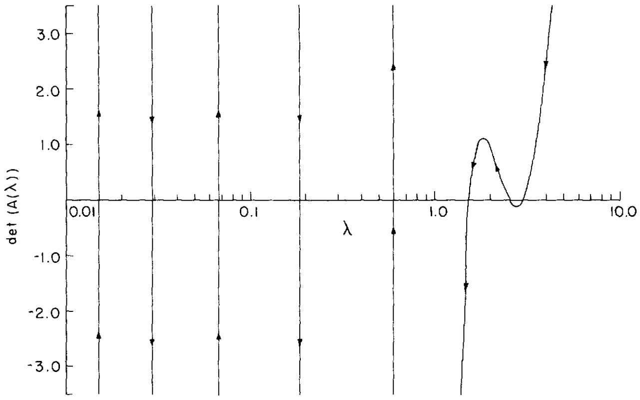

Fig. 2. det $(A(\lambda))$ vs λ.

all of the eigenvalues conveniently. In all cases that we have studied, we could account for at least 95% (often as much as 99%) of the total energy given by

$$\int_0^T \mathrm{Tr}[K_{\underline{y}}(t,t)]\, dt = E = \sum_{i=1}^{\infty} \lambda_i . \tag{84}$$

In Fig. 3 we have plotted the first six eigenvalues against T, the length of the interval ($T = T_f - T_o$). We see that the curves satisfy the monotonicity requirement,[1] and that the number of significant eigenvalues increases with T.

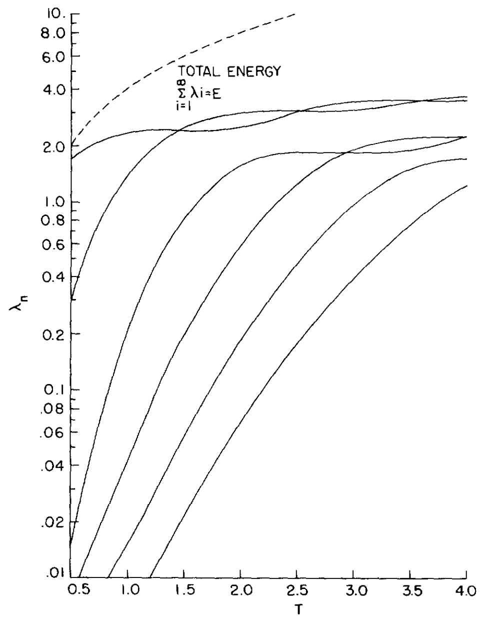

Fig. 3. λ_n vs T, $n = 1,2,\ldots,6$.

We shall conclude this example by discussing the computer time required to find the eigenvalues. We used the Fortran language on the IBM 7094 computer, and we did not employ any sophisticated algorithms for computing $A(\lambda)$. The time required to compute the data for Fig. 2 in order to find the first eight eigenvalues (99.8% of the energy) is approximately 20 seconds; in addition, the eigenfunctions may be found with very little additional computer time.

VI. NONHOMOGENEOUS FREDHOLM EQUATIONS

The nonhomogeneous Fredholm equation is given by Eq. 1

$$\int_{T_o}^{T_f} K_{\underline{y}}(t,\tau)\, \underline{g}(\tau)\, d\tau + \sigma \underline{g}(t) = \underline{s}(t), \qquad T_o \leq t \leq T_f.$$

In contrast to the homogeneous equation that has an at most countable number of solutions, this equation has a unique solution for $\sigma > 0$. We shall now employ the results of Section II to derive a set of differential equations and associated boundary conditions which specify $g(t)$, the solution to Eq. 1.

Let us rewrite Eq. 1. (This part of the derivation is due to a suggestion by L. D. Collins.[14]) By using Eq. 11, we have

$$\underline{g}(t) = \frac{1}{\sigma}\left\{\underline{s}(t) - C(t)\left(\int_{T_o}^{T_f} K_{\underline{x}}(t,\tau)\, C^T(\tau)\, \underline{g}(\tau)\, d\tau\right)\right\}, \quad T_o \leq t \leq T_f. \tag{85}$$

If in Eq. 85 we set

$$\underline{g}(t) = \underline{f}(t), \tag{86}$$

we have the result that the integral enclosed in parentheses is the function $\underline{\xi}(t)$ as defined in Eq. 18. Consequently, we define $\underline{\xi}(t)$ to be

$$\underline{\xi}(t) = \int_{T_o}^{T_f} K_{\underline{x}}(t,\tau)\, C^T(\tau)\, \underline{g}(\tau)\, d\tau, \qquad T_o \leq t \leq T_f, \tag{87}$$

so that Eq. 85 becomes

$$\underline{g}(t) = \frac{1}{\sigma}\left[\underline{s}(t) - C(t)\, \underline{\xi}(t)\right], \qquad T_o \leq t \leq T_f. \tag{88}$$

For the class of $K_{\underline{y}}(t,\tau)$ that we are considering, however, we have shown in Section II that the functional defined by Eq. 87 may be represented as the solution to the following differential equations:

$$\frac{d\underline{\xi}(t)}{dt} = F(t)\, \underline{\xi}(t) + G(t)\, Q\, G^T(t)\, \underline{\eta}(t), \qquad T_o \leq t \leq T_f \tag{89}$$

$$\frac{d\underline{\eta}(t)}{dt} = -C^T(t)\, \underline{g}(t) - F^T(t)\, \underline{\eta}(t), \qquad T_o \leq t \leq T_f \tag{90}$$

plus a set of boundary conditions. If we substitute Eq. 88 in Eq. 90, we obtain

$$\frac{d\underline{\eta}(t)}{dt} = \frac{C^T(t)\, C(t)}{\sigma}\, \underline{\xi}(t) - F^T(t)\, \underline{\eta}(t) - \frac{C^T(t)}{\sigma}\, \underline{s}(t), \quad T_o \leq t \leq T_f. \tag{91}$$

Consequently, we have shown that the nonhomogeneous Fredholm equation can be reduced to the following set of differential equations and associated boundary conditions:

$$\frac{d\underline{\xi}(t)}{dt} = F(t)\,\underline{\xi}(t) + G(t)\,Q\,G^T(t)\,\underline{\eta}(t), \qquad T_o \leq t \leq T_f, \tag{91}$$

$$\frac{d\underline{\eta}(t)}{dt} = \frac{C^T(t)\,C(t)}{\sigma}\,\underline{\xi}(t) - F^T(t)\,\underline{\eta}(t) - \frac{C^T(t)}{\sigma}\,\underline{s}(t) \qquad T_o \leq t \leq T_f \tag{92}$$

$$\underline{\xi}(T_o) = P_o\,\underline{\eta}(T_o), \tag{93}$$

$$\underline{\eta}(T_f) = 0. \tag{94}$$

The desired solution is given by Eq. 88 to be

$$\underline{g}(t) = \frac{1}{\sigma}\,[\underline{s}(t) - C(t)\,\underline{\xi}(t)], \qquad T_o \leq t \leq T_f. \tag{88}$$

An intuitively pleasing result is that the structure of the equations defining the solution is identical to that of the interval estimator for the optimum smoothing problem.[15,16] This is consistent with the estimator-subtractor form of the optimum receiver for the detection of $\underline{s}(t)$ in colored noise.[1] As a result, we can employ the solution methods that are available for the smoothing problem in order to solve Eqs. 91-94. We shall now briefly sketch three of these methods.

The first method is simply a superposition of the homogeneous and particular solutions. Let $\underline{\xi}_p(t)$ and $\underline{\eta}_p(t)$ be solutions to Eqs. 91 and 92 which satisfy the initial conditions

$$\underline{\xi}_p(T_o) = \underline{0} \tag{95a}$$

$$\underline{\eta}_p(T_o) = \underline{0}. \tag{95b}$$

To this particular solution let us add a homogeneous solution. This has the general form

$$\begin{bmatrix} \underline{\xi}_h(t) \\ \hline \underline{\eta}_h(t) \end{bmatrix} = \Psi(t,T_o:-\sigma) \begin{bmatrix} \underline{\xi}_h(T_o) \\ \hline \underline{\eta}_h(T_o) \end{bmatrix}, \tag{96}$$

where $\Psi(t,T_o:-\sigma)$ is defined by Eq. 41 in Section IV. When we apply the boundary conditions in order to solve for $\underline{\xi}_h(T_o)$ and $\underline{\eta}_h(T_o)$, we find that the total solution for $T_o \leq t \leq T_f$ is

$$\underline{\xi}(t) = \underline{\xi}_p(t) - [\Psi_{\underline{\xi}\underline{\xi}}(t,T_o:-\sigma)P_o + \Psi_{\underline{\xi}\underline{\eta}}(t,T_o:-\sigma)]\,\underline{\eta}_h(T_o) \tag{97a}$$

$$\underline{\eta}(t) = \underline{\eta}_p(t) - [\Psi_{\underline{\eta}\underline{\xi}}(t,T_o:-\sigma)P_o + \Psi_{\underline{\eta}\underline{\eta}}(t,T_o:-\sigma)]\,\underline{\eta}_h(T_o), \tag{97b}$$

where

$$\underline{\eta}_h(T_o) = [\Psi_{\underline{\eta}\underline{\xi}}(T_f,T_o:-\sigma)P_o + \Psi_{\underline{\eta}\underline{\eta}}(T_f,T_o:-\sigma)]^{-1}\,\underline{\eta}_p(T_f), \tag{97c}$$

and

$$\begin{bmatrix} \underline{\xi}_p(t) \\ \hline \underline{\eta}_p(t) \end{bmatrix} = -\int_0^t \Psi(t,\tau:-\sigma) \begin{bmatrix} 0 \\ \hline \dfrac{C^T(\tau)}{\sigma}\, \underline{s}(\tau) \end{bmatrix} d\tau \qquad T_o \leq t \leq T_f \tag{97d}$$

In order that a solution exist, the inverse indicated in Eq. 83c must exist. This existence can be shown for $\sigma > 0$.[10,17] If $\sigma < 0$ and the inverse did not exist, $-\sigma$ would be an eigenvalue of the homogeneous equation (cf. Eqs. 52 and 53).

The second method is related to the realizable filter structure as formulated by Kalman and Bucy.[10] In this method, one finds a complete set of boundary conditions at the end-point time, T_f, of the interval. Then the equations are solved backwards from this end point as if it were an initial value problem.

By paralleling techniques presented by Bryson and Frazier[15] or Baggeroer,[16] it can be shown that

$$\underline{\xi}(T_f) = \underline{\xi}_r(t)\Big|_{t=T_f}, \tag{98}$$

where $\underline{\xi}_r(t)$ satisfies the following set of differential equations and initial conditions

$$\frac{d\underline{\xi}_r(t)}{dt} = F(t)\, \underline{\xi}_r(t) + P(t)\, \frac{C^T(t)}{\sigma}\, [\underline{s}(t) - C(t)\, \underline{\xi}_r(t)] \tag{99a}$$

$$\frac{dP(t)}{dt} = F(t)\, P(t) + P(t)\, F^T(t) - P(t)\, \frac{C^T(t)\, C(t)}{\sigma}\, P(t) + G(t)\, Q\, G^T(t) \tag{99b}$$

$$\underline{\xi}_r(T_o) = \underline{0}, \qquad P(T_o) = P_o. \tag{99c-d}$$

The solution for $\underline{\xi}(t)$ over the entire interval follows directly by integrating Eqs. 91-92 backwards from the end point, $t = T_f$, or by applying the formula

$$\begin{bmatrix} \underline{\xi}(t) \\ \hline \underline{\eta}(t) \end{bmatrix} = \Psi(t,T_f:-\sigma) \begin{bmatrix} \underline{\xi}(T_f) \\ \hline \underline{0} \end{bmatrix} + \int_t^{T_f} \Psi(t,\tau:-\sigma) \begin{bmatrix} \underline{0} \\ \hline \dfrac{C^T(\tau)}{\sigma}\, \underline{s}(\tau) \end{bmatrix} d\tau. \tag{100}$$

The final method of solution is related to the formulation of the interval estimation problem by Rauch, Tung, and Striebel.[19] One can show that

$$\underline{\xi}(t) - \underline{\xi}_r(t) = P(t)\, \underline{\eta}(t), \qquad T_o \leq t \leq T_f, \tag{101}$$

By using this result, it can be shown that $\underline{\xi}(t)$ satisfies the differential equation

$$\frac{d\underline{\xi}(t)}{dt} = F(t)\, \underline{\xi}(t) + G(t)\, Q\, G^T(t)\, P^{-1}(t)\, (\underline{\xi}(t) - \underline{\xi}_r(t)), \tag{102}$$

where $\underline{\xi}(T_f)$ is given by Eq. 98. Alternatively, one can also show that $\underline{\eta}(t)$ satisfies the differential equation

$$\frac{d\underline{\eta}(t)}{dt} = \left[-F^T(t) + \frac{C^T(t)\ C(t)}{\sigma}\ P(t)\right] \underline{\eta}(t) - \frac{C^T(t)}{\sigma}\ [\underline{s}(t) - C(t)\underline{\xi}_r(t)], \tag{103}$$

where $\underline{\eta}(t)$ vanishes at $t = T_f$, and $\underline{\xi}(t)$ follows from Eq. 101. Consequently, by solving either Eq. 102 or 103 backwards in time, we may determine $\underline{\xi}(t)$. We should comment that this last method has some computational stability advantages.

The most significant advantage of all of these solution techniques is their easy implementation on a digital computer.

The solution indicated by Eq. 88 has an appealing interpretation when one calculates the d^2 performance measure for the detection of a known signal, $\underline{s}(t)$, in the presence of additive colored noise. The measure may be found from[1,6]

$$d^2 = \int_{T_o}^{T_f} \underline{s}^T(t)\ \underline{g}(t)\ dt. \tag{104}$$

Substituting Eq. 88 gives us

$$d^2 = \int_{T_o}^{T_f} \frac{\underline{s}^T(t)\ \underline{s}(t)}{\sigma}\ dt - \int_{T_o}^{T_f} \frac{\underline{s}^T(t)\ C(t)\ \underline{\xi}(t)}{\sigma}\ dt. \tag{105}$$

The first term is simply the pure white noise performance, d_w^2. The second term represents the degradation, d_g^2, caused by the presence of colored noise in the observation. Therefore, we have

$$d^2 = d_w^2 - d_g^2 \tag{106a}$$

$$d_w^2 = \int_{T_o}^{T_f} \left(\frac{\underline{s}^T(t)\ \underline{s}(t)}{\sigma}\right) dt \triangleq \frac{E}{\sigma} \tag{106b}$$

$$d_g^2 = \int_{T_o}^{T_f} \left(\frac{\underline{s}^T(t)\ C(t)\ \underline{\xi}(t)}{\sigma}\right) dt. \tag{106c}$$

VII. EXAMPLES OF THE SOLUTION TO THE NONHOMOGENEOUS EQUATION

We shall now present two examples of finding solutions to the nonhomogeneous equation, by using the results of Section VI. In the first example, we shall find analytic solutions for first-order systems; in the second, we shall determine a solution to a second-order system by using numerical methods.

Example 4

As in Section V, let us consider our kernel to be the covariance of a stationary process which is the output of a first-order system. Equations 67 and 68 describe a state representation for this system. For our signal, s(t), let us choose

$$s(t) = \sqrt{\frac{2}{T}}\,\sin\left(\frac{n\pi t}{T}\right) \qquad 0 \leq t \leq T, \tag{107}$$

corresponding to a sine wave with unit energy over the interval. From (91) and (92), the differential equations that we need to solve are

$$\frac{d}{dt}\left[\begin{array}{c}\xi(t)\\ \hline \eta(t)\end{array}\right] = \left[\begin{array}{c|c}-k & 2kS\\ \hline 1/\sigma & k\end{array}\right]\left[\begin{array}{c}\xi(t)\\ \hline \eta(t)\end{array}\right] - \left[\begin{array}{c}0\\ \hline \frac{1}{\sigma}\sqrt{\frac{2}{T}}\,\sin\left(\frac{n\pi t}{T}\right)\end{array}\right], \tag{108}$$

subject to the boundary conditions specified by Eqs. 93 and 94. We shall now use the first solution method which we discussed. After some manipulation, we find

$$\xi_p(t) = \sqrt{\frac{2}{T}}\,\frac{\gamma}{\sigma}\left(2kS\,\sin\left(\frac{n\pi t}{T}\right) - \frac{S}{\Lambda}\,\frac{n\pi}{T}\left(e^{k\Lambda t} - e^{-k\Lambda t}\right)\right) \tag{109a}$$

$$\eta_p(t) = \sqrt{\frac{2}{T}}\,\frac{\gamma}{\sigma}\left(k\,\sin\left(\frac{n\pi t}{T}\right) + \frac{n\pi}{T}\cos\left(\frac{n\pi t}{T}\right) - \frac{n\pi}{2T}\left(1 - \frac{1}{\Lambda}\right)e^{k\Lambda t} + \frac{n\pi}{2T}\left(1 + \frac{1}{\Lambda}\right)e^{-k\Lambda t}\right), \tag{109b}$$

where

$$\Lambda = \left[1 + \frac{2S}{k\sigma}\right]^{1/2} \tag{109c}$$

$$\gamma = \left[\left(\frac{n\pi}{T}\right)^2 + k^2\Lambda^2\right]^{-1}. \tag{109d}$$

In this example, the transition matrix, $\Psi(t,T_o:-\sigma)$, which we need is

$$\Psi(t,0:-\sigma) = \left[\begin{array}{c|c}\frac{1}{2}\left(1-\frac{1}{\Lambda}\right)e^{k\Lambda t} + \frac{1}{2}\left(1+\frac{1}{\Lambda}\right)e^{-k\Lambda t} & \frac{S}{\Lambda}e^{k\Lambda t} - \frac{S}{\Lambda}e^{-k\Lambda t}\\ \hline \frac{1}{2k\Lambda\sigma}e^{k\Lambda t} - \frac{1}{2k\Lambda t}e^{-k\Lambda t} & \frac{1}{2}\left(1+\frac{1}{\Lambda}\right)e^{k\Lambda t} + \frac{1}{2}\left(1-\frac{1}{\Lambda}\right)e^{-k\Lambda t}\end{array}\right]. \tag{110}$$

Therefore, according to Eq. 97, we have for $0 \leq t \leq T$

$$\xi(t) = \sqrt{\frac{2}{T}}\,\frac{\gamma}{\sigma}\left(2kS\,\sin\left(\frac{n\pi t}{T}\right) - \frac{S}{\Lambda}\frac{n\pi}{T}\left(e^{k\Lambda t}-e^{-k\Lambda t}\right)\right) \tag{111a}$$

$$-\left(\frac{S}{2}\left(1+\frac{1}{\Lambda}\right)e^{k\Lambda t}+\frac{S}{2}\left(1-\frac{1}{\Lambda}\right)e^{-k\Lambda t}\right)\eta_h(0)$$

$$\eta(t) = \sqrt{\frac{2}{T}}\,\frac{\gamma}{\sigma}\left(k\,\sin\left(\frac{n\pi t}{T}\right) + \frac{n\pi}{T}\cos\left(\frac{n\pi t}{T}\right) - \frac{n\pi}{2T}\left(1-\frac{1}{\Lambda}\right)e^{k\Lambda t} - \frac{n\pi}{2T}\left(1+\frac{1}{\Lambda}\right)e^{-k\Lambda t}\right)$$

$$-\left(\frac{(\Lambda+1)^2}{4\Lambda}e^{k\Lambda t} - \frac{(\Lambda-1)^2}{4\Lambda}e^{-k\Lambda t}\right)\eta_h(T_o), \tag{111b}$$

where

$$\eta_h(0) = \left(\frac{(\Lambda+1)^2}{4\Lambda}e^{k\Lambda T} - \frac{(\Lambda-1)^2}{4\Lambda}e^{-k\Lambda T}\right)^{-1}\sqrt{\frac{2}{T}}\,\frac{\gamma}{\sigma}\,\frac{n\pi}{T}\left((-1)^n - \frac{1}{2}\left(1+\frac{1}{\Lambda}\right)e^{k\Lambda T} - \frac{1}{2}\left(1-\frac{1}{\Lambda}\right)e^{-k\Lambda T}\right). \tag{111c}$$

Finally, using Eq. 88, we find the solution $g(t)$ for $0 \leq t \leq T$

$$g(t) = \sqrt{\frac{2}{T}}\,\frac{\gamma}{\sigma}\left\{\left[\left(\frac{n\pi}{T}\right)^2+k^2\right]\sin\left(\frac{n\pi t}{T}\right) - \frac{S}{\sigma}\frac{n\pi}{\Lambda T}\left(\frac{(\Lambda+1)^2}{2\Lambda}e^{k\Lambda T} - \frac{(\Lambda-1)^2}{2\Lambda}e^{-k\Lambda T}\right)^{-1}\right.$$

$$\left.\times\left((\Lambda+1)\left((-1)^n e^{k\Lambda t} - e^{-k\Lambda(T-t)}\right) + (\Lambda-1)\left((-1)^n e^{-k\Lambda t} - e^{-k\Lambda(T-t)}\right)\right)\right\}. \tag{112}$$

The complexity of this simple example illustrates the desirability of having a general numerical method available. One can continue and evaluate the performance by computing d^2 according to Eq. 101; however, the result is rather complex and is not too illustrative. Instead of presenting an analytic formula, we shall plot d^2 and d_g^2 against n for a particular choice of parameters. The results are presented in Fig. 4 when $k = 1$, $\sigma = 1$, $S = 1$, and $T = 2$. For the case $n = 1$, we see that the presence of the colored noise degrades our performance approximately 50% from that of the white noise. For $n = 8$, however, our performance is within 2% of the performance for the white noise only performance.

Example 5

For the second example, we shall consider a numerical approach to the analysis of a second-order system. In particular, we choose $K_y(t,\tau)$ to be the same as in Section V, Example 3. We also choose $\sigma = 1$, $T = 2$ and the signal, $s(t)$, to be

$$s(t) = \sqrt{\frac{2}{T}}\sin\left(\frac{n\pi}{T}t\right), \qquad 0 \leq t \leq T. \tag{113}$$

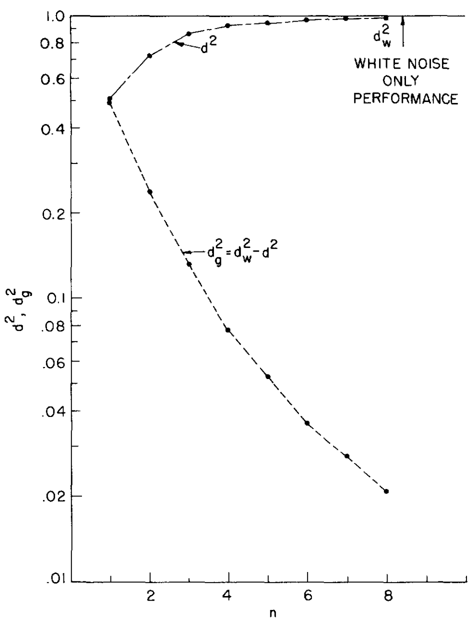

Fig. 4. d^2 and d_g^2 vs n for a first-order covariance function, $s(t) = \sqrt{2/T}\ \sin\ (n\pi t/T)$.

We have employed the first method discussed in Section VI, in order to find $\xi(t)$ and the solution g(t). We simply integrated the differential equation in increments of .01 in order to find the particular solutions $\xi_p(t)$ and $\eta_p(t)$. Then we added the

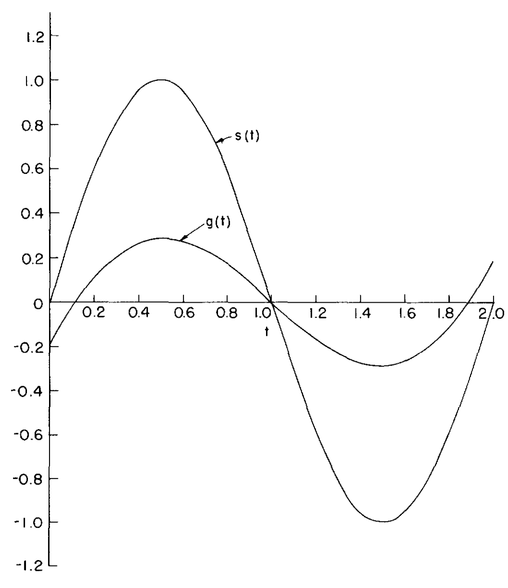

Fig. 5. g(t) and s(t) for a second-order covariance function, $s(t) = \sin\ (\pi t)$.

appropriate homogeneous solution, which we found by using the same matrix exponential that we used for the homogeneous equation.

In Figs. 5 and 6 we have drawn the signal s(t) and the corresponding solution g(t) for n = 2 and n = 8. For the low-frequency (n = 2) case, we find that functionally s(t) and g(t)

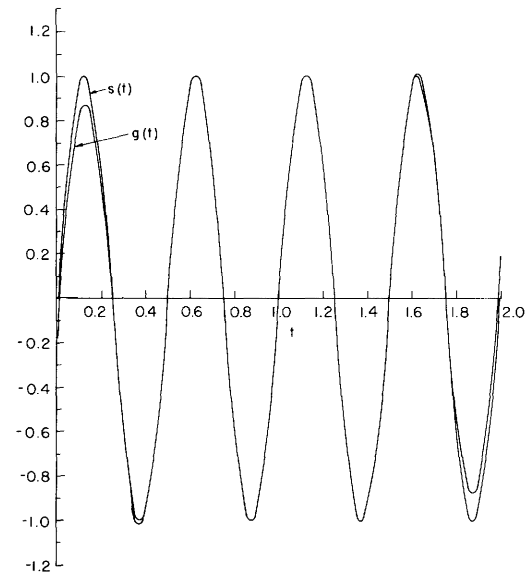

Fig. 6. g(t) and s(t) for a second-order covariance function, s(t) = sin (4πt).

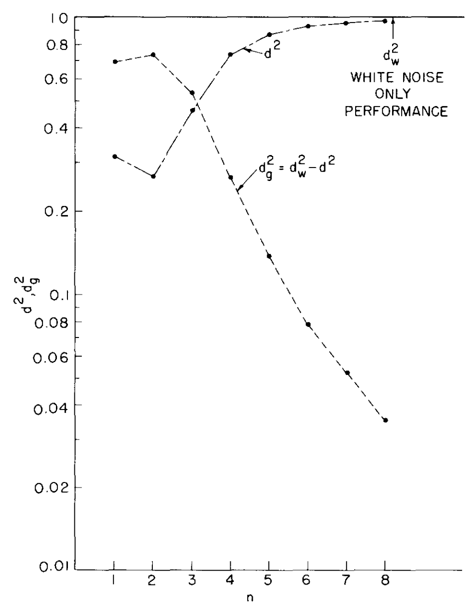

Fig. 7. d^2 and d_g^2 vs n for a second-order covariance function, $s(t) = \sqrt{2/T}\ \sin(n\pi t/T)$.

differ significantly only at the end points of the interval, while for the high-frequency ($n=8$) case we find that $s(t)$ and $g(t)$ are nearly identical. Here, we are approaching the white noise, or matched-filter, solution. We have summarized the results for this example in Fig. 7 by plotting the d^2 vs n behavior. We see that for $n > 8$ we are within 4% of the white noise only performance.

The computer time that we required to find a solution $g(t)$ and to calculate its performance is approximately 5-10 sec of IBM 7094 computer time, using the Fortran language. We expect that, by improved programming techniques, one could obtain a 50% reduction of this figure. To do this example analytically would be extremely tedious.

VIII. THE OPTIMUM LINEAR UNREALIZABLE FILTER

We shall now derive the state-variable realization of the optimum linear unrealizable filter by using results that we obtained in Sections III and VI. Although this realization is well known, we want to discuss it because the approach in our derivation is different from those in use at the present time.

Let us assume that we generate a random process $\underline{y}(t)$ by the methods described in Section II. Let us also assume that we observe this process in the presence of an additive white noise $\underline{w}(t)$ over the interval $T_o \leq t \leq T_f$. That is, we receive the signal

$$\underline{r}(t) = \underline{y}(t) + \underline{w}(t) = C(t)\ \underline{x}(t) + \underline{w}(t), \qquad T_o \leq t \leq T_f, \tag{114}$$

where $\underline{y}(t)$ is the output of a linear system that has a white noise input; $\underline{x}(t)$ is the state vector of this linear system; and $\underline{w}(t)$ is the white observation noise. The covariance of this additive white noise is

$$E[\underline{w}(t)\ \underline{w}^T(\tau)] = R(t)\ \delta(t-\tau), \tag{115}$$

where we assume that $R(t)$ is positive definite.

In the state-variable formulation of the optimum unrealizable filtering problem we want to find a state-variable description of the linear system that minimizes the mean-square error in estimating each component of the state vector $\underline{x}(t)$. We shall find that this description consists of two first-order vector differential equations having a two-point boundary restriction.

As we indicated previously, we use results from previous sections in our derivations. These results, however, are useful only when the received signal is a scalar and the additive noise is stationary, that is,

$$E[w(t)\ w(\tau)] = \sigma\ \delta(t-\tau). \tag{116}$$

We shall treat only this case in our derivation. The modifications of our previous results which would allow us to treat vector signals with time-varying noise level is straightforward.

The Wiener-Hopf equation[1] that specifies $\underline{h}(t,\tau)$, the impulse response of the optimum filter for estimating $\underline{x}(t)$, is

$$K_{\underline{x}}(t,\tau)\ C^T(\tau) = \int_{T_o}^{T_f} \underline{h}(t,\nu)\ K_r(\nu,\tau)\ d\nu, \qquad T_o \leq t,\tau \leq T_f, \tag{117}$$

where

$$K_r(t,\tau) = K_y(t,\tau) + \sigma\ \delta(t-\tau). \tag{118}$$

We now want to solve this equation for $\underline{h}(t,\tau)$. In order to do this, we need to introduce the inverse kernel $Q_r(t,\tau)$ of $K_r(t,\tau)$, the covariance of the received signal.

Our discussion is brief; for more detail see Van Trees.[1]

We can consider that the nonhomogeneous integral equation specifies a linear operator. The solution g(t) is the result of a linear operation upon s(t). Let us define the integral representation of this operation to be

$$g(t) = \int_{T_o}^{T_f} Q_r(t,\tau)\, s(\tau)\, d\tau, \qquad T_o \le t \le T_f. \tag{119}$$

Operating upon s(t) with $Q_r(t,\tau)$ to find g(t) is equivalent to solving the integral equation by means of our differential equation approach.

If we substitute (119) in the original integral equation (1), we obtain

$$\begin{aligned} s(t) &= \int_{T_o}^{T_f} Q_r(t,\tau)\, s(\tau)\, d\tau + \int_{T_o}^{T_f} \left(\int_{T_o}^{T_f} K_y(t,\nu)\, Q_r(\nu,\tau)\, d\nu \right) s(\tau)\, d\tau \\ &= \int_{T_o}^{T_f} \left(\int_{T_o}^{T_f} K_r(t,\nu)\, Q_r(\nu,\tau)\, d\nu \right) s(\tau)\, d\tau, \qquad T_o \le t \le T_f, \end{aligned} \tag{120}$$

where we brought the first term inside the integral by using a delta function, and we then used Eq. 118. Since s(t) is arbitrary, this implies

$$\int_{T_o}^{T_f} K_r(t,\nu)\, Q_r(\nu,\tau)\, d\nu = \delta(t-\tau), \qquad T_o \le t,\tau \le T_f, \tag{121}$$

which is the integral equation that the inverse kernel satisfies.

Let us multiply both sides of Eq. 117 by $Q_r(\tau,z)$ and then integrate with respect to τ. This yields

$$\begin{aligned} \int_{T_o}^{T_f} K_{\underline{x}}(t,\tau)\, C^T(\tau)\, Q_r(\tau,z)\, d\tau &= \int_{T_o}^{T_f} \underline{h}(t,\nu) \int_{T_o}^{T_f} K_r(\nu,\tau)\, Q_r(\tau,z)\, d\tau d\nu \\ &= \int_{T_o}^{T_f} h(t,\nu)\, \delta(\nu-z)\, d\nu = \underline{h}(t,z), \qquad T_o \le t,z \le T_f. \end{aligned} \tag{122}$$

We are not directly interested in the optimum impulse response; we want to find the estimate, $\hat{\underline{x}}(t)$, which is output of the optimum filter. This is given by

$$\hat{\underline{x}}(t) = \int_{T_o}^{T_f} \underline{h}(t,\tau)\, r(\tau)\, d\tau, \qquad T_o \le t \le T_f. \tag{123}$$

If we substitute Eq. 122 in Eq. 123, we obtain

$$\hat{\underline{x}}(t) = \int_{T_o}^{T_f} K_{\underline{x}}(t,\tau)\, C^T(\tau) \left(\int_{T_o}^{T_f} Q_r(\tau,z)\, r(z)\, dz \right) d\tau, \qquad T_o \le t \le T_f. \tag{124}$$

We now want to show how we can reduce Eq. 124 to two differential equations with an associated set of boundary conditions. Let us define the term in parentheses in Eq. 124 as $g_r(\tau)$, so that we have

$$g_r(\tau) = \int_{T_o}^{T_f} Q_r(\tau,z)\, r(z)\, dz, \qquad T_o \le \tau \le T_f \tag{125a}$$

and

$$\hat{\underline{x}}(t) = \int_{T_o}^{T_f} K_{\underline{x}}(t,\tau)\, C^T(\tau)\, g_r(\tau)\, d\tau, \qquad T_o \le t \le T_f. \tag{125b}$$

Observe that (125a) and (125b) are integral operations of the type encountered in Sections VI and III, respectively. Consequently, we can convert each into two vector differential equations with an associated set of boundary conditions.

From our previous discussion, $g_r(\tau)$ is the solution to the nonhomogeneous Fredholm integral equation when the signal $s(\tau)$ is replaced by $r(\tau)$. From Section VI, we have

$$g_r(\tau) = \frac{1}{\sigma}\left(r(\tau) - C(\tau)\, \underline{\xi}(\tau)\right), \tag{126}$$

where $\underline{\xi}(\tau)$ is the solution to the differential equations,

$$\frac{d\underline{\xi}(\tau)}{d\tau} = F(\tau)\, \underline{\xi}(t) + G(\tau)\, Q\, G^T(\tau)\, \underline{\eta}_1(\tau), \tag{127a}$$

$$\frac{d\underline{\eta}_1(\tau)}{d\tau} = \frac{C^T(\tau)\, C(\tau)}{\sigma}\, \underline{\xi}(\tau) - F^T(\tau)\, \underline{\eta}_1(\tau) - \frac{C^T(\tau)}{\sigma}\, r(\tau). \tag{127b}$$

The boundary conditions are

$$\underline{\xi}(T_o) = P_o\, \underline{\eta}_1(T_o) \tag{128a}$$

$$\underline{\eta}_1(T_f) = \underline{0}. \tag{128b}$$

In Section III we found that the operation given by Eq. 125b also has a differential equation representation. If in Eq. 17 we set

$$f(\tau) = g_r(\tau), \tag{129}$$

the optimum estimate $\hat{\underline{x}}(t)$ may be found from the differential equations (Eqs. 25 and 27):

$$\frac{d\hat{\underline{x}}(t)}{dt} = F(t)\, \hat{\underline{x}}(t) + G(t)\, Q\, G^T(t)\, \underline{\eta}_2(t) \tag{130a}$$

$$\begin{aligned} \frac{d\underline{\eta}_2(t)}{dt} &= -\, C^T(t)\, g_r(t) - F^T(t)\, \underline{\eta}_2(t) \\ &= \frac{C^T(t)\, C(t)}{\sigma}\, \underline{\xi}(t) - F^T(t)\, \underline{\eta}_2(t) - \frac{C^T(t)}{\sigma}\, r(t). \end{aligned} \tag{130b}$$

The boundary condtions for these equations (Eqs. 30 and 28) are

$$\hat{\underline{x}}(T_o) = P_o\ \underline{\eta}_2(T_o) \tag{131a}$$

$$\underline{\eta}_2(T_f) = \underline{0}. \tag{131b}$$

At first glance, we need to solve four vector differential equations with an associated set of boundary conditions. But, if we examine Eq. 127b and Eq. 130b, we find that $\underline{\eta}_1(t)$ and $\underline{\eta}_2(t)$ satisfy the same differential equation. Since both equations have the same boundary condition at $t = T_f$ (Eqs. 128b and 131b), they must have identical solutions. Consequently, we have

$$\underline{\eta}_1(t) = \underline{\eta}_2(t) \triangleq \underline{\eta}(t), \qquad T_o \leq t \leq T_f. \tag{132}$$

Now, by replacing $\underline{\eta}_1(t)$ and $\underline{\eta}_2(t)$ by $\underline{\eta}(t)$ in Eqs. 127a, 128a, 130a, and 131a, we see that $\underline{\xi}(t)$ and $\hat{\underline{x}}(t)$ satisfy the same differential equations (Eqs. 127a and 130a) and have the same boundary conditions (Eqs. 128a and 131a). Therefore, we must have

$$\hat{\underline{x}}(t) = \underline{\xi}(t), \qquad T_o \leq t \leq T_f. \tag{133}$$

Consequently, we have shown that two of the four differential equations are redundant.

We finally obtain the state-variable representation of the linear optimum unrealizable filter. The optimum estimate satisfies the differential equations

$$\frac{d\hat{\underline{x}}(t)}{dt} = F(t)\ \hat{\underline{x}}(t) + G(t)\ Q\ G^T(t)\ \underline{\eta}(t) \tag{134a}$$

$$\frac{d\underline{\eta}(t)}{dt} = \frac{C^T(t)\ C(t)}{\sigma}\ \hat{\underline{x}}(t) - F^T(t)\ \underline{\eta}(t) - \frac{C^T(t)}{\sigma}\ r(t). \tag{134b}$$

The boundary conditions are

$$\hat{\underline{x}}(T_o) = P_o\ \underline{\eta}(T_o) \tag{135a}$$

$$\underline{\eta}(T_f) = \underline{0}. \tag{135b}$$

As we mentioned before, the realization specified by Eqs. 134 and 135 is well known. Our method differs from existing ones, in that it is a structured approach. We have required the filter structure to be linear, regardless of the statistics of the processes involved. Existing approaches to this problem are unstructured; it is assumed that the processes involved are Gaussian and then the filter structure is derived. It is well known, however, that both approaches yield the same filter.

Finally integral equations similar to the Wiener-Hopf equation that appears in (117) are found in the general Gaussian detection problem. Techniques similar to those used in this section are also applicable to solving these integral equations.

IX. SUMMARY AND CONCLUSIONS

We have presented a state-variable technique for solving Fredholm integral equations of the second kind. We have emphasized that the principal advantage of the technique is that it leads to efficient computer algorithms for actually obtaining numerical solutions. We have also pointed out several other advantages of this technique.

We developed the technique by finding a differential equation representation for a particular linear operator that appeared in both the homogeneous and nonhomogeneous integral equations. This representation allowed us to reduce both integral equations to two first-order linear differential equations plus an associated set of boundary conditions.

In the case of the homogeneous integral equation, we found that we could determine both the eigenvalues and eigenfunctions in terms of the transition matrix of the vector differential equations. The eigenvalues were related to the determinant of a matrix that was computed in terms of partitions of this transition matrix. Similarly, the eigenfunctions were also found in terms of partitions of this same transition matrix.

We then discussed several examples which illustrated both analytic methods and the results of numerical analysis. In our studies we have analyzed systems up to fifth order with very little difficulty.

The differential equations that we obtained in the case of the nonhomogeneous equation were identical to those that have been obtained in the state-variable formulation of the optimum linear smoother. We briefly discussed three methods of solution for this problem, which were applicable to the solution of these differential equations. We then considered two examples that were presented in the context of detecting a known signal in the presence of additive colored noise.

Finally, we used a new approach for finding the state-variable realization of the optimum linear unrealizable filter (smoother). Our approach was a structured approach. We required our filter to be linear. Then by using the resulting Wiener-Hopf equation in conjuction with techniques developed in this report, we were able to derive equations that specified this minimum mean-square estimator.

One issue that we did omit was that of finding the state equations for generating the kernel of the integral equation. For the class of kernels which may be generated by using constant-parameter systems, this is straightforward. Quite often, however, the needed state-variable description may be part of the original data rather than the actual kernel of the integral equations.

There are several areas in communication theory to which the techniques and results developed in this report are applicable. We shall mention three of them.

The first area is the calculation of performance measures. We have a convenient method for finding numerically the eigenvalues that appear in Karhunen-Loève expansion theory; therefore, we may calculate performance measures of many systems in terms of these eigenvalues. Typically, these measures may be probabilities of error in detection problems or error variances in parameter estimation studies.[21]

A second area is optimal signal design. By using Pontryagin's Maximum Principle in conjunction with our differential equations, we may study various optimization problems. In the problem of detecting a known signal in additive colored noise that we discussed, we have found a method of optimizing the signal s(t) so as to maximize the d^2 performance when we constrain its energy and bandwidth.[11,18,20] It is possible to extend the method to include other types of constraints, for example, peak value of the signal.

The third area is realizable filtering with delay. If we allow a delay before making an estimate, we can often attain significant improvement in the accuracy of our estimate. We can apply the results that we obtained for the linear unrealizable filter (fixed interval) to derive the state-variable formulation of the filter which is realizable with delay. It is also possible to find differential equations that specify its performance.

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13. ABSTRACT

A method of solving Fredholm integral equations of the second kind by state-variable techniques is presented. The principal advantage of this method is that it leads to efficient computer algorithms for calculating numerical solutions. The only assumptions that are made are (a) the kernel of the integral equation is the covariance function of a random process; (b) this random process is the output of a linear system having a white noise input; (c) this linear system has a finite dimensional state-variable description of its input-output relationship.

Both the homogeneous and nonhomogeneous integral equations are reduced to two linear first-order vector differential equations plus an associated set of boundary conditions. The coefficients of these differential equations follow directly from the matrices that describe the linear system. In the case of the homogeneous integral equation, the eigenvalues are found to be the solutions to a transcendental equation. The eigenfunctions also follow directly. In the case of the nonhomogeneous equation, the vector differential equations are identical to those obtained in the state-variable formulation of the optimum linear smoother. In both cases analytical and numerical examples are presented.

Finally, the optimum linear smoother (unrealizable filter) structure is derived by using a new approach. In this approach, the filter is required to be linear; then the resulting Wiener-Hopf equation is used in conjunction with techniques developed in the report to find the differential equations and boundary conditions specifying the optimum estimate.

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